Exact Eqns Examples

1. Solve $(2x+3) + (2y-2)y' = 0$

Soln:

$(2x+3)\,dx + (2y-2)\,dy = 0$

$M = 2x+3$

$N = 2y-2$

Check if $M_y = N_x$

$$M_y = \partial_y M = \partial_y(2x+3) = 2 \qquad N_x = \partial_x N = \partial_x(2y-2) = 2$$

$M_y = N_x$

wt $M = \partial_x F$ and $N = \partial_y F$

$\partial_x F = M$

$$F = \int M\,dx = \int 2x+3\,dx = x^2 + 3x + c(y)$$

$N = \partial_y F$

$$2y - 2 = \partial_y(x^2 + 3x + c(y)) = c'(y)$$

$$c = \int 2y - 2\,dy = y^2 - 2y + c'$$

wt $c' = 0$

$F = x^2 + 3x + y^2 - 2y = C$

2. Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

Soln:

$(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$

$M = 3x^2 - 2xy + 2$

$N = 6y^2 - x^2 + 3$

Check if $M_y = N_x$

$M_y = \partial_y M$ $\quad$ $N_x = \partial_x N$

$= -2x$ $\quad$ $= -2x$

$M_y = N_x$

Let $M = \partial_x f$ and $N = \partial_y f$

$\partial_x f = M$

$f = \int M\,dx$

$= \int 3x^2 - 2xy + 2\,dx$

$= x^3 - x^2 y + 2x + c(y)$

$N = \partial_y f$

$= \partial_y (x^3 - x^2 y + 2x + c(y))$

$6y^2 - x^2 + 3 = -x^2 + c'(y)$

$c'(y) = 6y^2 + 3$

$c = \int 6y^2 + 3\,dy$

$= 2y^3 + 3y + c_1$

Let $c_1 = 0$

$F = x^3 - x^2 y + 2x + 2y^3 + 3y = 0$

3. Find an integrating factor for and solve
$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$

Soln:
$(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0$

$M = 3x^2y + 2xy + y^3$
$N = x^2 + y^2$

To determine if μ, the integrating factor, is a function of x, check if $\frac{M_y - N_x}{N}$ contains "y".

$$\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2}$$
$$= \frac{3(x^2 + y^2)}{x^2 + y^2}$$
$$= 3$$

Since $\frac{M_y - N_x}{N}$ does not contain y, μ is a function of x.

Note: To determine if μ is a function of y, see if $\frac{N_x - M_y}{M}$ contains "x".

$$\frac{N_x - M_y}{M} = \frac{2x - (3x^2 + 2x + 3y^2)}{3x^2y + 2xy + y^3}$$

$$= \frac{-3(x^2 + y^2)}{3x^2y + 2xy + y^3}$$ ← Cannot simplify any more and contains x. Hence, μ is not a function of y.

$$\frac{\mu'}{\mu} = 3$$

$(\ln(\mu))' = 3$

$\int (\ln(\mu))' \, dx = \int 3 \, dx$

$\ln(\mu) + c_1 = 3x + c_2$

$\ln(\mu) = 3x + c$

$\mu = e^{3x+c}$

$\quad = e^{c} \cdot e^{3x}$

Let $e^{c} = 1$

$\mu = e^{3x}$

We multiply both sides of the original eqn by μ.

$$e^{3x}(3x^2y + 2xy + y^3)\,dx + e^{3x}(x^2 + y^2)\,dy = 0$$

$M_1 = 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x}$

$N_1 = x^2e^{3x} + y^2e^{3x}$

Let $\partial_x F = M_1$ and $\partial_y F = N_1$

$\partial_x F = M_1$

$F = \int M_1 \, dx$

$\quad = \int 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x} \, dx$

$$= \frac{y(9x^2 - 6x + 2)e^{3x}}{9} + \frac{2y(3x-1)e^{3x}}{9} + \frac{y^3e^{3x}}{3} + c(y)$$

$$= \frac{(3x^2y + y^3)e^{3x}}{3} + c(y)$$

$N_1 = \partial y F$

$$x^2e^{3x} + y^2e^{3x} = \partial y\left(\frac{(3x^2y + y^3)e^{3x}}{3} + c(y)\right)$$

$$= \partial y\left(x^2ye^{3x} + \frac{y^3e^{3x}}{3} + c(y)\right)$$

$$= \partial y(x^2ye^{3x}) + \partial y\left(\frac{y^3e^{3x}}{3}\right) + \partial y(c(y))$$

$$= x^2e^{3x} + y^2e^{3x} + c'(y)$$

$$c'(y) = 0$$

$$c = 0$$

$$F = \frac{(3x^2y + y^3)e^{3x}}{3} = c$$

4. Find an integrating factor for and solve $y + (2xy - e^{-2y})y' = 0$.

Soln:

$$y\,dx + (2xy - e^{-2y})\,dy = 0$$

$M = y \qquad N = 2xy - e^{-2y}$

Check if $\frac{M_y - N_x}{N}$ contains y or not.

$$\frac{M_y - N_x}{N} = \frac{1 - \quad 2y}{2xy - e^{-2y}}$$

← Can no longer simplify and it contains y. Hence, μ is not a function of x.

Check if $\frac{N_x - M_y}{M}$ contains x or not.

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y}$$
$$= 2 - \frac{1}{y}$$ ← Does not contain x. Hence, μ is a function of y.

$$\frac{\mu'}{\mu} = 2 - \frac{1}{y}$$
$$(\ln(\mu))' = 2 - \frac{1}{y}$$
$$\int (\ln(\mu))' \, dy = \int 2 - \frac{1}{y} \, dy$$
$$\ln(\mu) + C_1 = 2y - \ln|y| + C_2$$
$$\ln(\mu) = 2y - \ln|y| + c$$
$$\mu = e^{2y - \ln|y| + c}$$
$$= \frac{e^{2y}}{y}$$

Multiply both sides of the original eqn by μ.

$$\frac{e^{2y}}{y} y \, dx + \frac{e^{2y}}{y}\left(2xy - e^{-2y}\right) dy = 0$$

$$e^{2y} \, dx + (2xe^{2y} - 1/y) \, dy = 0$$
$$M_1 = e^{2y} \qquad N_1 = 2xe^{2y} - 1/y$$

Let $\partial_x F = M_1$ and $\partial_y F = N_1$

$\partial_x F = M_1$

$F = \int M_1 \, dx$

$= \int e^{2y} \, dx$

$= xe^{2y} + C(y)$

$N_1 = \partial_y F$

$2xe^{2y} - \frac{1}{y} = \partial_y (xe^{2y} + C(y))$

$= 2xe^{2y} + C'(y)$

$C'(y) = -\frac{1}{y}$

$C = \int -\frac{1}{y} \, dy$

$= -\ln|y| + C_1$

Let $C_1 = 0$

$C = -\ln|y|$

$F = xe^{2y} - \ln|y| = C$